Proof Writing & Verifications

Sliding Along

Problem

Prove $\exists(x \in \mathbb{R}).\ \forall(y \in \mathbb{R}).\ x > y$.

Proof

Let $x = y + 1$. Clearly $y + 1 > y$.

Commentary

Note that y is never introduced. So add "Let $y \in \mathbb{R}$ be arbitrary" at the beginning to fix this. Then, note that the theorem this *actually* proves is $\forall(y \in \mathbb{R}).\ \exists(x \in \mathbb{R}).\ x > y$. The order in which we introduce variables corresponds to the order of their quantifiers in the theorem we prove! More generally, not introducing variables properly hides the structure of your proof.

Verifications

Verifications are debugging for proofs. When most people first learn to program, they are forced to pick up debugging as a skill, because it's so common. Think of Verifications as "debugging for proofs." It may not be a glamorous skill to practice, but it's definitely a necessary skill have.

Verification Goals. Recall the major goals for verifications:

- Understand the problem (and solution to the problem) that you're verifying
- Understand the possible mistakes you could have made on *any proof on a similar topic*
- Get better at reading and analyzing arguments written by other people
- Get better at analyzing your own arguments for correctness

Verifications for the future.

- Many of you will go to Graduate School. You'll need to be able to read other peoples' proofs in Academia.
- Even if you go to industry, some of your co-workers will make algorithmic or math-based arguments, and you'll be expected to understand them and fix them if necessary.

Same Difference

Here, we provide two versions of the same proof. Our goal here is to convince you that the slightly more verbose proof is actually significantly better.

Problem

Prove that for any sets, A and B, $A \cup B = A \cup (B \setminus A)$.

Proof 1

Suppose $x \in A \cup B$ and $x \notin A \cup (B \setminus A)$. Then $x \notin A$ so $x \in B$. But $x \notin B \setminus A$ so $x \notin B$ or $x \in A$! Otherwise suppose $x \notin A \cup B$ and $x \in A \cup (B \setminus A)$. Then $x \notin A$ so $x \in B \setminus A$ so $x \in B$. But $x \notin B$. So, we are done.

Proof 2

We show for any sets A and B that $A \cup B = A \cup (B \setminus A)$. Consider any sets A and B. Assume for contradiction that $A \cup B \neq A \cup (B \setminus A)$. Then, there must be some element x that is in one of $A \cup B$ or $A \cup (B \setminus A)$ but not the other. Consider this x. By cases:

- Suppose $x \in A \cup B$ but $x \notin A \cup (B \setminus A)$. Either $x \in A$ or $x \notin A$, so by cases:
  - Suppose $x \in A$. Then $x \in A \cup (B \setminus A)$, which contradicts our assumption.
  - Suppose $x \notin A$. Since by assumption $x \in A \cup B$, either $x \in A$ or $x \in B$, so since $x \notin A$, we must have $x \in B$. But then $x \in B$ and $x \notin A$, so $x \in B \setminus A$, so $x \in A \cup (B \setminus A)$, which contradicts our assumption.
- Suppose $x \notin A \cup B$ but $x \in A \cup (B \setminus A)$. Since $x \notin A \cup B$, we have $x \notin A$ and $x \notin B$. Since $x \notin A$ but $x \in A \cup (B \setminus A)$, we must have $x \in B \setminus A$. So by the definition of $\setminus$ we have $x \in B$ and $x \notin A$. But we already showed that $x \notin B$, so we have a contradiction.

Since each case leads to a contradiction, our assumption is false, and $A \cup B = A \cup (B \setminus A)$. This proves the claim.

Demo Verification 1: Induction

Problem

Recall that f_n is the n-th Fibonacci number. Show that f_{3k} is even for all $k \in \mathbb{N}$.

Proof

We go by induction to prove that this is true (14).
Base Case: $f_0 = 0$ is even.
Induction Hypothesis: Assume the claim holds (3) for $k \in \mathbb{N}$.
Induction Step: Choose n (12) such that $f_k = 2n$ (2). Then

$$\begin{aligned} f_{3k+3} &= f_{3k+2} + f_{3k+1} \\ &= f_{3k+1} + f_{3k} + f_{3k+1} \\ &= 2f_{3k+1} + f_{3k} \\ &= 2(f_{3k+1} + n) \end{aligned}$$

Therefore, f_{3k+3} is even.

General

1 ( ) The submission uses induction informally (i.e. it says "and keep going...")

2 (×) The proof contains a typo that changes the meaning of the proof.

Statement Definition

3 (×) The proof refers to a statement $P(n)$ or "the claim" without ever defining it.

Base Case

4 ( ) A necessary base case is missing or incorrect.

5 ( ) The proof has no base cases at all.

6 ( ) The proof has at least one base case, but the one(s) there don't cover what is needed by the Induction Hypothesis.

Induction Hypothesis

7 ( ) The Induction Hypothesis is missing.

8 ( ) The Induction Hypothesis assumes the conclusion of the proof.

Induction Step

9 ( ) The Induction Step does not use the same variable defined by the Induction Hypothesis.

10 ( ) The Induction Step makes a minor arithmetic error.

11 ( ) The proof fails to quantify a variable in the Induction Step.

12 (×) The proof does not explicitly invoke the Induction Hypothesis.

Induction Style

13 ( ) The proof does not indicate which variable is being inducted on.

14 ($\times$) The proof does not provide any justifications for any step of the mathematical derivation.

Demo Verification 2: Bijection

Problem

Let $f : [2, \infty) \to [-3, \infty)$ be defined by $f(x) = x^2 - 4x + 1$. Prove that f is a bijection.

Proof

Injection: Let $x, y \in [2, \infty)$. Assume (14)

$$
\begin{aligned}
& & f(x) &= f(y) && \text{[assumption]} \\
& \Longrightarrow & x^2 - 4x + 1 &= y^2 - 4y + 1 && \text{[definition]} \\
& \Longrightarrow & x^2 - 4x &= y^2 - 4y && \text{[algebra]} \\
& \Longrightarrow & x^2 - y^2 &= 4x - 4y && \text{[algebra]} \\
& \Longrightarrow & (x+y)(x-y) &= 4(x-y) && \text{[algebra]} \\
\text{(4)} & \Longrightarrow & x + y &= 4 && \text{[algebra]}
\end{aligned}
$$

Since $x, y \geq 2$ and $x + y = 4$, this is only possible when $x = y = 2$. Thus, $x = y$. $\therefore$ (18) f is injective.
Surjection: Let x (10), y (7) $\in [2, \infty)$.

$$\text{(6)}\ x = 2 + \sqrt{3+y}$$

$$f\left(2+\sqrt{3+y}\right) = \left(2+\sqrt{3+y}\right)^2 - 4\left(2+\sqrt{3+y}\right) + 1 = 4 + 4\sqrt{3+y} + 3 + y - 8 + \text{(1)}\sqrt{3+y} + 1 = \text{(15)}\,y$$

Thus, $\exists$ an x such that $f(x) = y$. $\therefore$ (18) f is surjective. (16) [Missing conclusion that f is bijection.]

General

1 ($\times$) The proof contains a typo that changes the meaning of the proof.

Injection

2 ( ) The proof incorrectly cites or uses the definition of injection.

3 ( ) The proof tries to prove the implication without assuming the hypothesis.

4 ($\times$) The proof divides by zero.

Some Judgements that don't apply are missing.

Surjection

5 ( ) The proof incorrectly cites or uses the definition of surjection.

6 ($\times$) The surjection proof does not justify why the chosen value in the domain is actually in the domain.

7 ($\times$) The proof specifies the wrong domain for a variable.

8 ( ) The surjection proof starts by setting $y = f(x)$ and finding a value, but it doesn't justify why this is reversible.

9 ( ) The proof chooses a value for a variable when it should have let it be arbitrary.

10 ($\times$) The proof lets a variable be arbitrary when it should have just chosen a single value.

Some Judgements that don't apply are missing.

Inverse

11 ( ) The proof finds an incorrect inverse function.

12 ( ) The proof only proves one of the two equalities necessary to show that a function is an inverse.

13 ( ) The proof does not explain why the existence of an inverse actually shows that the function is a bijection.

Some Judgements that don't apply are missing.

Style

14 ($\times$) The proof starts a chain of equations with "let" or "assume" (this does not make sense).

15 ($\times$) The proof doesn't justify every step well.

16 ($\times$) The proof lacks at least one conclusion.

17 ( ) The proof does not provide any justifications for any step of the mathematical derivation.

18 ($\times$) The proof mixes mathematical symbols and words in sentences.

19 ( ) The proof confuses equals signs with implication (or bi-implication) arrows.