CS 13: Mathematics for Computer Scientists

Verification Homework 00 (due Friday, April 7)

Directions: *In this assignment, you will review and "judge" purported proofs corresponding to a claim claim. We provide you with a "reference solution" for each claim before asking you to verify another solution.*

Claim

Define f_n and g_n as follows for $n \in \mathbb{N}$:

$$\begin{aligned} f_0 &= 1 & g_0 &= 1 \\ f_1 &= 5 & g_1 &= 5 \\ f_2 &= 10 & g_2 &= 10 \\ & & g_3 &= 0 \\ & & g_4 &= -40 \\ f_n &= 2f_{n-1} - 4f_{n-2} \text{ for } n \geq 3 & g_n &= 2g_{n-1} - 3g_{n-2} - 2g_{n-3} + 4g_{n-4} \text{ for } n \geq 5 \end{aligned}$$

Prove that $f_n = g_n$ for all $n \in \mathbb{N}$.

Reference Solution

We will show that $f_n = g_n$ for all $n \in \mathbb{N}$ by strong induction on n.

Base Cases. For $n = 0$, $n = 1$, and $n = 2$, the equality of f_n and g_n follows directly from the definitions of f_n and g_n. For $n = 3$, $f_3 = 2f_2 - 4f_1 = 2(10) - 4(5) = 0 = g_3$, and for $n = 4$, we have $f_4 = 2f_3 - 4f_2 = 2(0) - 4(10) = -40 = g_4$ by the definitions of f and g.

Induction Hypothesis. Assume that for some $k \in \mathbb{N}$, $4 \leq k$, that the claim holds for all $l \in \mathbb{N}$, $0 \leq l \leq k$.

Induction Step.

$$\begin{aligned} g_{k+1} &= 2g_k - 3g_{k-1} - 2g_{k-2} + 4g_{k-3} && \text{[Definition of } g_n \text{ for } n \geq 5.] \\ &= 2f_k - 3f_{k-1} - 2f_{k-2} + 4f_{k-3} && \text{[Induction Hypothesis.]} \\ &= 2f_k - 3f_{k-1} - f_{k-1} && \text{[Definition of } f_n \text{ for } n \geq 3.] \\ &= 2f_k - 4f_{k-1} && \text{[Algebra.]} \\ &= f_{k+1} && \text{[Definition of } f_n \text{ for } n \geq 3.] \end{aligned}$$

Since the base cases and the induction step hold, $f_n = g_n$ for all $n \in \mathbb{N}$.

Student Proof 0

Directions: *Read the proof on this page and carefully determine any syntactic or semantic errors it might make. On judgements rubric on the next page, you will fill in which errors you believe this proof has in it.*

We are given that f_n and g_n is defined for all $n \in \mathbb{N}$ as follows:

$$f_0 = 1 \qquad g_0 = 1$$
$$f_1 = 5 \qquad g_1 = 5$$
$$f_2 = 10 \qquad g_2 = 10$$
$$g_3 = 0$$
$$g_4 = -40$$
$$\vdots \qquad \vdots$$
$$f_n = 2f_{n-1} - 4f_{n-2} \text{ for } n \geq 3 \qquad g_n = 2g_{n-1} - 3g_{n-2} - 2g_{n-3} + 4g_{n-4} \text{ for } n \geq 5$$

Let $n \in \mathbb{N}$ be arbitrary and fixed.
Let $P(n)$ be "$f_n = g_n$". We will prove $P(n)$ is true for all $n \in \mathbb{N}$ by induction on n.
Base Cases: Show $P(0)$ is true: $1 = 1$ is true and therefore $P(0)$ holds.
Show $P(1)$ is true: $5 = 5$ is true and therefore $P(1)$ holds.
Show $P(2)$ is true: $10 = 10$ is true and therefore $P(2)$ holds.
Show $P(3)$ is true: $0 = 0$ is true and therefore $P(3)$ holds.
Show $P(4)$ is true: $-40 = -40$ is true and therefore $P(4)$ holds.
From the above descriptions we have shown that $P(0)$, $P(1)$, $P(2)$, $P(3)$, and $P(4)$ all hold true and therefore the base cases hold true for the statement $P(n)$
Induction Hypothesis: Assume $P(0) \wedge P(1) \wedge ... \wedge P(k)$ is true for some $k \in \mathbb{N}$, and we also say that $k \geq 5$ because we have already shown $P(n)$ holds for the cases up to 5.
Induction Step: We will prove $P(k) \implies P(k+1)$ by proving that P(k+1) holds. To do this we want to show:
$f_{k+1} = g_{k+1}$
$2f_k - 4f_{k-1} = 2g_k - 3g_{k-1} - 2g_{k-2} + 4g_{k-3}$ [Substitution from the Given]
By the induction hypothesis we assumed $P(k)$, $P(k-1)$, $P(k-2)$, $P(k-3)$ are true and therefore from this we can assume $f_k = g_k$, $f_{k-1} = g_{k-1}$, $f_{k-2} = g_{k-2}$,$f_{k-3} = g_{k-3}$ are true and can substitute these values in for g on the right side to get a new equality:
$2f_k - 4f_{k-1} = 2f_k - 3f_{k-1} - 2f_{k-2} + 4f_{k-3}$
$-f_{k-1} = -2f_{k-2} + 4f_{k-3}$ [Algebra]
$f_{k-1} = 2f_{k-2} - 4f_{k-3}$ [Multiplication by -1 on both sides] Let $m = k - 1$ and then we can rewrite this final statement: $f_m = 2f_{m-1} - 4f_{m-2}$
And this is in the form of the given that $f_n = 2f_{n-1} - 4f_{n-2}$ when $n = m$ and therefore because this is given as always true, we have shown that $P(k+1)$ is also true.
Thus by induction using the base cases, the induction hypothesis, and the induction step, we have shown that $f_n = g_n$ is true for all $n \in \mathbb{N}$

Judgements for Proof 0

Directions: *For each "judgement" in this rubric, determine if it applies anywhere to the above proof. If it does apply, check the box and note the number on the proof itself where the error occurs.*

Writing Style

1 ( ) Cites that steps are "trivial", "obvious" or "by definition" when they require justification

2 ( ) Omits a justification that is an application of a theorem or definition

3 ( ) The proof does not have a conclusion

4 ( ) Uses implication arrows between steps

5 ( ) Mixes mathematical symbols and words in sentences

6 ( ) Confuses equals signs with implication (or bi-implication) arrows

Well-defining Variables

7 ( ) Uses a variable without declaration or quantification

8 ( ) Variable declaration or quantification does not match the declaration/quantification in the claim.

9 ( ) Quantifies a variable multiple times (both "arbitrary" and "a specific thing")

10 ( ) Instantiates variables in the wrong order (or makes a variable dependent on an undefined variable)

11 ( ) Chooses a set of values for an existential instead of a single one

12 ( ) Starts a chain of equations with "let" or "assume" (this does not make sense)

13 ( ) Lacks at least one conclusion

14 ( ) The proof does not provide any justifications for any step of the mathematical derivation

Logical Errors

15 ( ) Typo in the proof changes the meaning of the proof.

16 ( ) The proof contains inconsistent or undefined notation (e.g., uses $f(n)$ and f_n interchangeably without defining notation).

17 ( ) The proof misuses mathematical symbols (e.g., $\in$ instead of $\subseteq$).

18 ( ) Proof makes an assumption without explicitly stating that assumption.

19 ( ) Several consecutive steps of the proof are unrelated or unconnected.

20 ( ) Starts from the conclusion and arrives at something which is true instead of going forward. (This is "backwards reasoning".)

21 ( ) Attempts to prove something false (instead of proving something true)

22 ( ) Proves the converse of the statement instead of the intended statement.

23 ( ) Lacks proofs in both directions for an if-and-only-if.

24 ( ) Uses proof by contradiction when it is unnecessary (this usually happens when the contradiction is the claim they were trying to prove).

25 ( ) $\implies$ does not use two truth statements (e.g., a set definition instead of declaring membership in that set).

Induction

26 ( ) Uses $P(n)$ or refers to "the claim" without defining it.

27 ( ) Uses a function as a statement or the reverse (e.g. proves "$P(n) = n + 1$" by induction on n)

28 ( ) Has a mismatch between variables in the definition of $P(n)$ or the IH.

29 ( ) Induction step increments the wrong variable in strong induction.

30 ( ) Too few (but at least one) base case(s) are given to establish the induction hypothesis.

31 ( ) The proof includes an unnecessary base case.

32 ( ) The induction hypothesis states "assume $P(k)$" without defining k (either before or after)

33 ( ) The induction hypothesis quantifies at least one variable incorrectly.

34 ( ) The induction hypothesis assumes the conclusion of the proof.

35 ( ) The induction hypothesis does not say "assume," "suppose," or something similar.

36 ( ) The induction hypothesis assumes $P(k) \implies P(k+1)$ instead of just $P(k)$.

37 ( ) The induction step assumes the conclusion (independently of the induction hypothesis).

38 ( ) The proof does not explicitly invoke the induction hypothesis.

39 ( ) The induction step invokes the induction hypothesis in a part of the argument unrelated to the induction hypothesis.

Student Proof 1

Directions: *Read the proof on this page and carefully determine any syntactic or semantic errors it might make. On judgements rubric on the next page, you will fill in which errors you believe this proof has in it.*

Let $P(n)$ be the statement "$f(n) = g(n)$". We prove $P(n)$ for all $n \in \mathbb{N}$ by strong induction.

Base Cases:

- $P(0)$ is true because by the definition of $f(n)$ and $g(n)$, $f_0 = 1, g_0 = 1$ and therefore $f_0 = g_0$.
- $P(1)$ is true because by the definition of $f(n)$ and $g(n)$, $f_1 = 5, g_1 = 5$ and therefore $f_1 = g_1$.
- $P(2)$ is true because by the definition of $f(n)$ and $g(n)$, $f_2 = 10, g_2 = 10$ and therefore $f_2 = g_2$.
- $P(3)$ is true because by the definition of $f(n)$, $f_3 = 2f_2 - 4f_1$. Since we know $f_2 = 10$ and $f_1 = 5$, we know that $f_3 = 2f_2 - 4f_1 = 2(10) - 4(5) = 0$. We also know that $g_3 = 0$. Therefore, $f_3 = g_3$.
- $P(4)$ is true because by the definition of $f(n)$, $f_4 = 2f_3 - 4f_2$. Since we know $f_2 = 10$ and $f_3 = 0$, we know that $f_4 = 2f_3 - 4f_2 = 2(0) - 4(10) = -40$. We also know that $g_4 = -40$. Therefore, $f_4 = g_4$.

Induction Hypothesis:
Suppose that, for some $k \in \mathbb{N}$ and $l \in \mathbb{N}$, $P(k)$ is true for all $k \leq l$ such that $k \geq 0, l \geq x$.
Induction Step:
By the definition of f_n for $n \geq 3$, we know that $f_{l+1} = 2f_l - 4fl - 1$. We can reason that

$$\begin{aligned}
2f_l - 4f_{l-1} &= 2f_l - 3f_{l-1} - f_{l-1} && \text{[break up } 4f_{l-1}\text{]} \\
&= 2f_l - 3f_{l-1} - (2f_{l-2} - 4f_{l-3}) && \text{[definition of } f_n\text{]} \\
&= 2f_l - 3f_{l-1} - 2f_{l-2} + 4f_{l-3} && \text{[distribution]} \\
&= 2g_l - 3g_{l-1} - 2g_{l-2} + 4g_{l-3} && \text{[Induction Hypothesis]}
\end{aligned}$$

This is precisely g_{l+1}, so $P(l + 1)$ is true.

Since the base case and the induction step hold, the claim holds true for all $n \in \mathbb{N}$.

Judgements for Proof 1

Directions: *For each "judgement" in this rubric, determine if it applies anywhere to the above proof. If it does apply, check the box and note the number on the proof itself where the error occurs.*

Writing Style

1 ( ) Cites that steps are "trivial", "obvious" or "by definition" when they require justification

2 ( ) Omits a justification that is an application of a theorem or definition

3 ( ) The proof does not have a conclusion

4 ( ) Uses implication arrows between steps

5 ( ) Mixes mathematical symbols and words in sentences

6 ( ) Confuses equals signs with implication (or bi-implication) arrows

Well-defining Variables

7 ( ) Uses a variable without declaration or quantification

8 ( ) Variable declaration or quantification does not match the declaration/quantification in the claim.

9 ( ) Quantifies a variable multiple times (both "arbitrary" and "a specific thing")

10 ( ) Instantiates variables in the wrong order (or makes a variable dependent on an undefined variable)

11 ( ) Chooses a set of values for an existential instead of a single one

12 ( ) Starts a chain of equations with "let" or "assume" (this does not make sense)

13 ( ) Lacks at least one conclusion

14 ( ) The proof does not provide any justifications for any step of the mathematical derivation

Logical Errors

15 ( ) Typo in the proof changes the meaning of the proof.

16 ( ) The proof contains inconsistent or undefined notation (e.g., uses $f(n)$ and f_n interchangeably without defining notation).

17 ( ) The proof misuses mathematical symbols (e.g., $\in$ instead of $\subseteq$).

18 ( ) Proof makes an assumption without explicitly stating that assumption.

19 ( ) Several consecutive steps of the proof are unrelated or unconnected.

20 ( ) Starts from the conclusion and arrives at something which is true instead of going forward. (This is "backwards reasoning".)

21 ( ) Attempts to prove something false (instead of proving something true)

22 ( ) Proves the converse of the statement instead of the intended statement.

23 ( ) Lacks proofs in both directions for an if-and-only-if.

24 ( ) Uses proof by contradiction when it is unnecessary (this usually happens when the contradiction is the claim they were trying to prove).

25 (   ) $\implies$ does not use two truth statements (e.g., a set definition instead of declaring membership in that set).

Induction

26 (   ) Uses $P(n)$ or refers to "the claim" without defining it.

27 (   ) Uses a function as a statement or the reverse (e.g. proves "$P(n) = n + 1$" by induction on n)

28 (   ) Has a mismatch between variables in the definition of $P(n)$ or the IH.

29 (   ) Induction step increments the wrong variable in strong induction.

30 (   ) Too few (but at least one) base case(s) are given to establish the induction hypothesis.

31 (   ) The proof includes an unnecessary base case.

32 (   ) The induction hypothesis states "assume $P(k)$" without defining k (either before or after)

33 (   ) The induction hypothesis quantifies at least one variable incorrectly.

34 (   ) The induction hypothesis assumes the conclusion of the proof.

35 (   ) The induction hypothesis does not say "assume," "suppose," or something similar.

36 (   ) The induction hypothesis assumes $P(k) \implies P(k+1)$ instead of just $P(k)$.

37 (   ) The induction step assumes the conclusion (independently of the induction hypothesis).

38 (   ) The proof does not explicitly invoke the induction hypothesis.

39 (   ) The induction step invokes the induction hypothesis in a part of the argument unrelated to the induction hypothesis.

Student Proof 2

Directions: *Read the proof on this page and carefully determine any syntactic or semantic errors it might make. On judgements rubric on the next page, you will fill in which errors you believe this proof has in it.*

Let f_n and g_n be defined as follows for $n \in \mathbb{N}$

$f_0 = 1,\ f_1 = 5,\ f_2 = 10,\ f_n = 2f_{n-1} - 4fn - 2$ for $n \geq 3$.

$g_0 = 1,\ g_1 = 5,\ g_2 = 10,\ g_3 = 0,\ g_4 = -40,\ g_n = 2g_{n-1} - 3g_{n-2} - 2g_{n-3} + 4g_{n-4}$ for $n \geq 5$.

I will prove by strong induction that $f_n = g_n$ for all $n \in \mathbb{N}$.

Base Cases:

$f_0 = 1 = g_0$

$f_1 = 5 = g_1$

$f_2 = 10 = g_2$

$f_3 = 2(10) - 4(5) = 0 = g_3$

$f_4 = 2(0) - 4(10) = -40 = g_4$

We can see that for $n \in \{0, 1, 2, 3, 4\}$, $f_n = g_n$. Therefore all of our base cases hold.

Induction Hypothesis:

Let x and z be natural numbers. Assume for all $(z < x)$,

$f_z = g_z \implies f_x = g_x$

Induction Step:

$$
\begin{aligned}
& g_x \\
&= 2g_{x-1} - 3g_{x-2} - 2g_{x-3} + 4g_{x-4} && \text{[By def of } g_n] \\
&= 2f_{x-1} - 3f_{x-2} - 2f_{x-3} + 4f_{x-4} && \text{[By our induction hypothesis]} \\
&= 2f_{x-1} - 3f_{x-2} - f_{x-2} && \text{[By def of } f_n] \\
&= 2f_{x-1} - 4f_{x-2} && \text{[Combining like terms]} \\
&= f_x && \text{[By def of } f_n]
\end{aligned}
$$

Because the base cases and induction step hold, we can conclude that $f_n = g_n$ for all n.

Judgements for Proof 2

Directions: *For each "judgement" in this rubric, determine if it applies anywhere to the above proof. If it does apply, check the box and note the number on the proof itself where the error occurs.*

Writing Style

1 ( ) Cites that steps are "trivial", "obvious" or "by definition" when they require justification

2 ( ) Omits a justification that is an application of a theorem or definition

3 ( ) The proof does not have a conclusion

4 ( ) Uses implication arrows between steps

5 ( ) Mixes mathematical symbols and words in sentences

6 ( ) Confuses equals signs with implication (or bi-implication) arrows

Well-defining Variables

7 ( ) Uses a variable without declaration or quantification

8 ( ) Variable declaration or quantification does not match the declaration/quantification in the claim.

9 ( ) Quantifies a variable multiple times (both "arbitrary" and "a specific thing")

10 ( ) Instantiates variables in the wrong order (or makes a variable dependent on an undefined variable)

11 ( ) Chooses a set of values for an existential instead of a single one

12 ( ) Starts a chain of equations with "let" or "assume" (this does not make sense)

13 ( ) Lacks at least one conclusion

14 ( ) The proof does not provide any justifications for any step of the mathematical derivation

Logical Errors

15 ( ) Typo in the proof changes the meaning of the proof.

16 ( ) The proof contains inconsistent or undefined notation (e.g., uses $f(n)$ and f_n interchangeably without defining notation).

17 ( ) The proof misuses mathematical symbols (e.g., $\in$ instead of $\subseteq$).

18 ( ) Proof makes an assumption without explicitly stating that assumption.

19 ( ) Several consecutive steps of the proof are unrelated or unconnected.

20 ( ) Starts from the conclusion and arrives at something which is true instead of going forward. (This is "backwards reasoning".)

21 ( ) Attempts to prove something false (instead of proving something true)

22 ( ) Proves the converse of the statement instead of the intended statement.

23 ( ) Lacks proofs in both directions for an if-and-only-if.

24 ( ) Uses proof by contradiction when it is unnecessary (this usually happens when the contradiction is the claim they were trying to prove).

25 (   ) $\implies$ does not use two truth statements (e.g., a set definition instead of declaring membership in that set).

Induction

26 (   ) Uses $P(n)$ or refers to "the claim" without defining it.

27 (   ) Uses a function as a statement or the reverse (e.g. proves "$P(n) = n + 1$" by induction on n)

28 (   ) Has a mismatch between variables in the definition of $P(n)$ or the IH.

29 (   ) Induction step increments the wrong variable in strong induction.

30 (   ) Too few (but at least one) base case(s) are given to establish the induction hypothesis.

31 (   ) The proof includes an unnecessary base case.

32 (   ) The induction hypothesis states "assume $P(k)$" without defining k (either before or after)

33 (   ) The induction hypothesis quantifies at least one variable incorrectly.

34 (   ) The induction hypothesis assumes the conclusion of the proof.

35 (   ) The induction hypothesis does not say "assume," "suppose," or something similar.

36 (   ) The induction hypothesis assumes $P(k) \implies P(k+1)$ instead of just $P(k)$.

37 (   ) The induction step assumes the conclusion (independently of the induction hypothesis).

38 (   ) The proof does not explicitly invoke the induction hypothesis.

39 (   ) The induction step invokes the induction hypothesis in a part of the argument unrelated to the induction hypothesis.

Student Proof 3

Directions: *Read the proof on this page and carefully determine any syntactic or semantic errors it might make. On judgements rubric on the next page, you will fill in which errors you believe this proof has in it.*

Claim: $f_n = g_n$ for all $n \in \mathbb{N}$

Let $P(n)$ be the statement "$f_n = g_n$".

We shall prove that $\forall(n \in \mathbb{N}).\ P(n)$ is true by strong induction on n.

Base Case:

$P(0)$ is true because $f_0 = 1 = g_0$.

$P(1)$ is true because $f_1 = 5 = g_1$.

$P(2)$ is true because $f_2 = 10 = g_2$.

$P(3)$ is true because $f_3 = 2f_2 - 4f_1 = 2(10) - 4(5) = 0 = g_3$.

$P(4)$ is true because $f_4 = 2f_3 - 4f_2 = 2(0) - 4(10) = -40 = g_4$.

Induction Hypothesis:

Suppose that $P(k)$ is true for all $k \leq l$ for some $k, l \in \mathbb{N}$ such that $k \geq 0, l \geq 4$.

Induction Step:

$$
\begin{aligned}
f_{k+1} &= 2f_k - 4f_{k-1} && \text{[Given]} \\
&= 2f_k - 3f_{k-1} - f_{k-1} && \text{[Algebra]} \\
&= 2f_k - 3f_{k-1} - (2f_{k-2} - 4f_{k-3}) && \text{[By IH]} \\
&= 2f_k - 3f_{k-1} - 2f_{k-2} + 4f_{k-3} && \text{[Simplify]} \\
&= 2g_k - 3g_{k-1} - 2g_{k-2} + 4g_{k-3} && \text{[By IH]} \\
&= g_{k+1}
\end{aligned}
$$

Thus, we have proven that $f_{k+1} = g_{k+1}$, which means that $P(l+1)$ holds.

Since the Base Case and the Induction Step hold, $P(n)$ is true for all natural numbers n.

Judgements for Proof 3

Directions: *For each "judgement" in this rubric, determine if it applies anywhere to the above proof. If it does apply, check the box and note the number on the proof itself where the error occurs.*

Writing Style

1 ( ) Cites that steps are "trivial", "obvious" or "by definition" when they require justification

2 ( ) Omits a justification that is an application of a theorem or definition

3 ( ) The proof does not have a conclusion

4 ( ) Uses implication arrows between steps

5 ( ) Mixes mathematical symbols and words in sentences

6 ( ) Confuses equals signs with implication (or bi-implication) arrows

Well-defining Variables

7 ( ) Uses a variable without declaration or quantification

8 ( ) Variable declaration or quantification does not match the declaration/quantification in the claim.

9 ( ) Quantifies a variable multiple times (both "arbitrary" and "a specific thing")

10 ( ) Instantiates variables in the wrong order (or makes a variable dependent on an undefined variable)

11 ( ) Chooses a set of values for an existential instead of a single one

12 ( ) Starts a chain of equations with "let" or "assume" (this does not make sense)

13 ( ) Lacks at least one conclusion

14 ( ) The proof does not provide any justifications for any step of the mathematical derivation

Logical Errors

15 ( ) Typo in the proof changes the meaning of the proof.

16 ( ) The proof contains inconsistent or undefined notation (e.g., uses $f(n)$ and f_n interchangeably without defining notation).

17 ( ) The proof misuses mathematical symbols (e.g., $\in$ instead of $\subseteq$).

18 ( ) Proof makes an assumption without explicitly stating that assumption.

19 ( ) Several consecutive steps of the proof are unrelated or unconnected.

20 ( ) Starts from the conclusion and arrives at something which is true instead of going forward. (This is "backwards reasoning".)

21 ( ) Attempts to prove something false (instead of proving something true)

22 ( ) Proves the converse of the statement instead of the intended statement.

23 ( ) Lacks proofs in both directions for an if-and-only-if.

24 ( ) Uses proof by contradiction when it is unnecessary (this usually happens when the contradiction is the claim they were trying to prove).

25 ( ) $\implies$ does not use two truth statements (e.g., a set definition instead of declaring membership in that set).

Induction

26 ( ) Uses $P(n)$ or refers to "the claim" without defining it.

27 ( ) Uses a function as a statement or the reverse (e.g. proves "$P(n) = n + 1$" by induction on n)

28 ( ) Has a mismatch between variables in the definition of $P(n)$ or the IH.

29 ( ) Induction step increments the wrong variable in strong induction.

30 ( ) Too few (but at least one) base case(s) are given to establish the induction hypothesis.

31 ( ) The proof includes an unnecessary base case.

32 ( ) The induction hypothesis states "assume $P(k)$" without defining k (either before or after)

33 ( ) The induction hypothesis quantifies at least one variable incorrectly.

34 ( ) The induction hypothesis assumes the conclusion of the proof.

35 ( ) The induction hypothesis does not say "assume," "suppose," or something similar.

36 ( ) The induction hypothesis assumes $P(k) \implies P(k+1)$ instead of just $P(k)$.

37 ( ) The induction step assumes the conclusion (independently of the induction hypothesis).

38 ( ) The proof does not explicitly invoke the induction hypothesis.

39 ( ) The induction step invokes the induction hypothesis in a part of the argument unrelated to the induction hypothesis.

Student Proof 4

Directions: *Read the proof on this page and carefully determine any syntactic or semantic errors it might make. On judgements rubric on the next page, you will fill in which errors you believe this proof has in it.*

To prove the statement, we will be using induction as the proof method.

Let $P(n)$ be the statement "$f_n = g_n$" for all $n \in \mathbb{N}$. We will prove $P(n)$ for all $n \in \mathbb{N}$ by induction on n.

Base Cases:

$P(0)$: f_0= 1 $= g_0$ by definition of sequence. $P(1)$: f_1= 5 $= g_1$ by definition of sequence. $P(2)$: f_2= 10 $= g_2$ by definition of sequence.

$P(3)$:

$$\begin{aligned} f_3 &= 2f_2 - 4f_1 && \text{[By definition of } f_n\text{]} \\ &= 0 && \text{[This is } g_3\text{. Therefore } f_3 = g_3\text{; } P(3) \text{ proven]} \end{aligned}$$

$P(4)$:

$$\begin{aligned} f_4 &= 2f_3 - 4f_2 && \text{[By definition of } f_n\text{]} \\ &= -40 && \text{[This is } g_4\text{. Therefore } f_4 = g_4\text{; } P(4) \text{ proven]} \end{aligned}$$

Induction Hypothesis: Suppose $P(k)$ for some k where $0 \leq k \leq l$ where $l \geq 4$.

Induction Step:

From the IH, we know $P(l), P(l-1), P(l-2), P(l-3)$.

Prove $P(l+1)$:

$$\begin{aligned} g_{l+1} &= 2g_l - 3g_{l-1} - 2g_{l-2} + 4g_{l-3} && \text{[By definition of } g_n\text{]} \\ &= 2f_l - 3f_{l-1} - 2f_{l-2} + 4f_{l-3} && \text{[By IH: } P(l), P(l-1), P(l-2), P(l-3)\text{]} \\ &= 2f_l - 3f_{l-1} - (2f_{l-2} - 4f_{l-3}) && \text{[Factorization]} \\ &= 2f_l - 3f_{l-1} - (f_{l-1}) && \text{[Definition of } f_n\text{]} \\ &= 2f_l - 4f_{l-1} && \text{[Algebra]} \\ &= f_{l+1} && \text{[Definition of } f_n\text{]} \\ g_{l+1} &= f_{l+1} && \text{[Induction Step Proven]} \end{aligned}$$

Therefore, since the base cases and the induction step hold, the claim holds for all $n \in \mathbb{N}$.

Judgements for Proof 4

Directions: *For each "judgement" in this rubric, determine if it applies anywhere to the above proof. If it does apply, check the box and note the number on the proof itself where the error occurs.*

Writing Style

1 ( ) Cites that steps are "trivial", "obvious" or "by definition" when they require justification

2 ( ) Omits a justification that is an application of a theorem or definition

3 ( ) The proof does not have a conclusion

4 ( ) Uses implication arrows between steps

5 ( ) Mixes mathematical symbols and words in sentences

6 ( ) Confuses equals signs with implication (or bi-implication) arrows

Well-defining Variables

7 ( ) Uses a variable without declaration or quantification

8 ( ) Variable declaration or quantification does not match the declaration/quantification in the claim.

9 ( ) Quantifies a variable multiple times (both "arbitrary" and "a specific thing")

10 ( ) Instantiates variables in the wrong order (or makes a variable dependent on an undefined variable)

11 ( ) Chooses a set of values for an existential instead of a single one

12 ( ) Starts a chain of equations with "let" or "assume" (this does not make sense)

13 ( ) Lacks at least one conclusion

14 ( ) The proof does not provide any justifications for any step of the mathematical derivation

Logical Errors

15 ( ) Typo in the proof changes the meaning of the proof.

16 ( ) The proof contains inconsistent or undefined notation (e.g., uses $f(n)$ and f_n interchangeably without defining notation).

17 ( ) The proof misuses mathematical symbols (e.g., $\in$ instead of $\subseteq$).

18 ( ) Proof makes an assumption without explicitly stating that assumption.

19 ( ) Several consecutive steps of the proof are unrelated or unconnected.

20 ( ) Starts from the conclusion and arrives at something which is true instead of going forward. (This is "backwards reasoning".)

21 ( ) Attempts to prove something false (instead of proving something true)

22 ( ) Proves the converse of the statement instead of the intended statement.

23 ( ) Lacks proofs in both directions for an if-and-only-if.

24 ( ) Uses proof by contradiction when it is unnecessary (this usually happens when the contradiction is the claim they were trying to prove).

25 ( ) $\implies$ does not use two truth statements (e.g., a set definition instead of declaring membership in that set).

Induction

26 ( ) Uses $P(n)$ or refers to "the claim" without defining it.

27 ( ) Uses a function as a statement or the reverse (e.g. proves "$P(n) = n + 1$" by induction on n)

28 ( ) Has a mismatch between variables in the definition of $P(n)$ or the IH.

29 ( ) Induction step increments the wrong variable in strong induction.

30 ( ) Too few (but at least one) base case(s) are given to establish the induction hypothesis.

31 ( ) The proof includes an unnecessary base case.

32 ( ) The induction hypothesis states "assume $P(k)$" without defining k (either before or after)

33 ( ) The induction hypothesis quantifies at least one variable incorrectly.

34 ( ) The induction hypothesis assumes the conclusion of the proof.

35 ( ) The induction hypothesis does not say "assume," "suppose," or something similar.

36 ( ) The induction hypothesis assumes $P(k) \implies P(k+1)$ instead of just $P(k)$.

37 ( ) The induction step assumes the conclusion (independently of the induction hypothesis).

38 ( ) The proof does not explicitly invoke the induction hypothesis.

39 ( ) The induction step invokes the induction hypothesis in a part of the argument unrelated to the induction hypothesis.