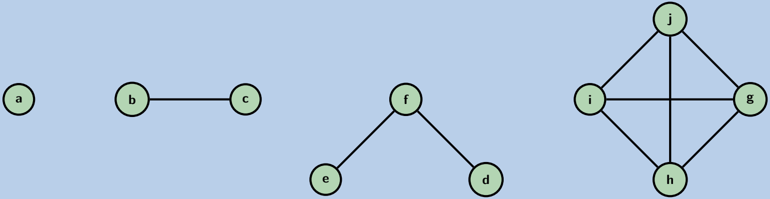
CS 13

Mathematical Foundations of Computing

Graphs



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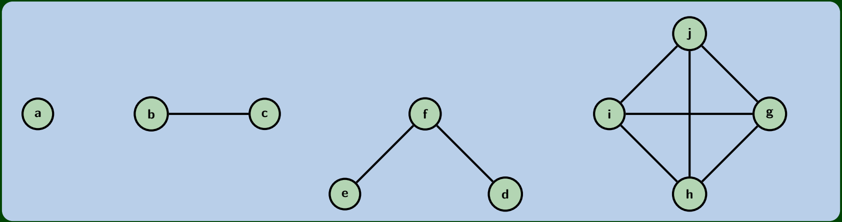

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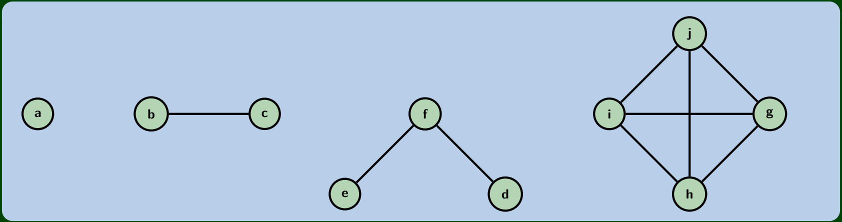

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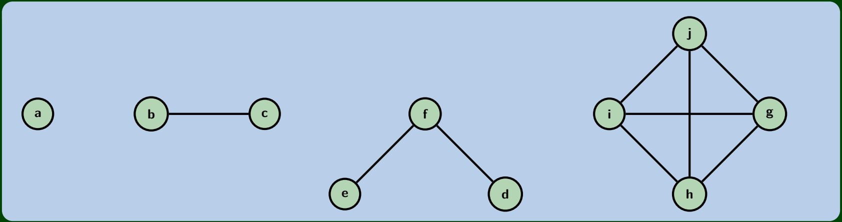

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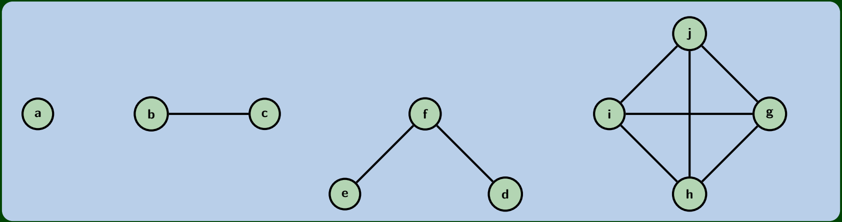

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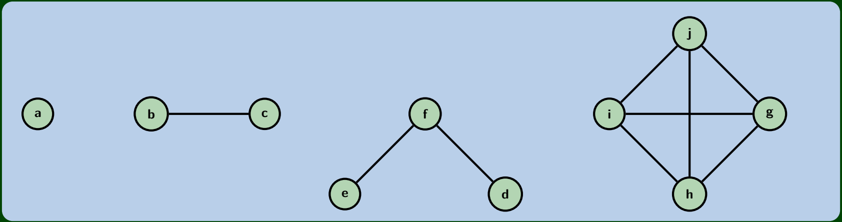

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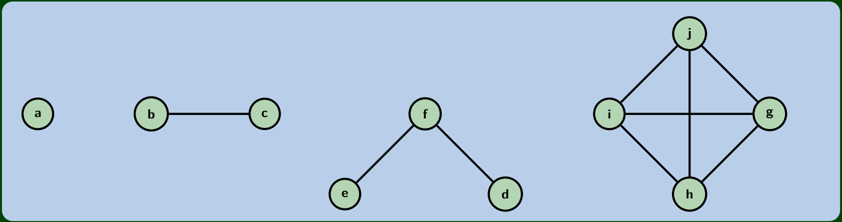

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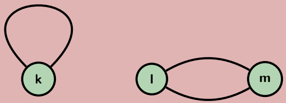
Formally, a graph $G = (V, E)$, where V is a set of vertices and E is a set of edges.

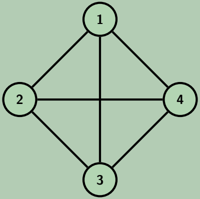
Not Graphs!



No graph (that we will look at) has any **loops** or **multiple edges**.

Definition (Complete Graph)

The complete graph on n vertices is called K_n. We define it as $K_n = ([n], \{\{x,y\} \mid x,y \in [n] \land x \neq y\})$. Earlier, we saw K_4:



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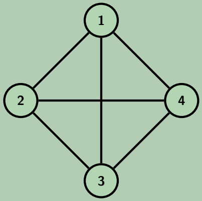

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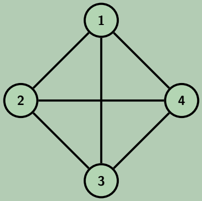

Question: How many edges does K_6 have?
Answer: Any two vertices can make an edge. So, there are $\binom{6}{2}$ pairs of vertices. Then, there are $\binom{6}{2} = 15$ possible edges in K_6.

Definition (Path)

Let G be a graph. We say that there is a path between two vertices $u, v \in V$ iff there is a list of edges $[\{u, x_1\}, \{x_1, x_2\}, \ldots, \{x_k, v\}]$ such that no vertex is hit twice, the first edge contains u, and the last edge contains v.

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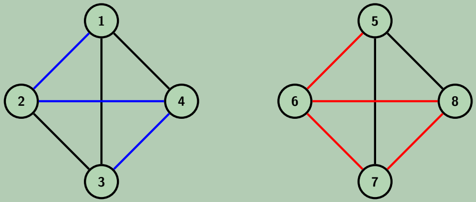
More intuitively, a path is a continuous line from u to v that doesn't repeat vertices. Pictures are helpful:



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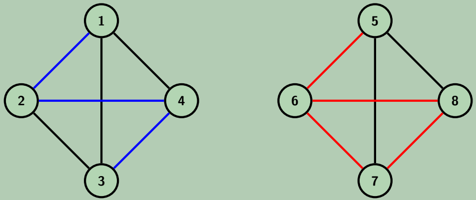

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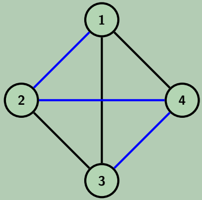

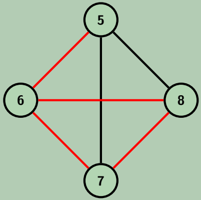

The blue edges $[\{1,2\},\{2,4\},\{4,3\}]$ indicate a **valid** path from 1 to 3. The red edges $[\{5,6\},\{6,8\},\{8,7\},\{7,6\}]$ indicate an **invalid** path from 5 to 6.

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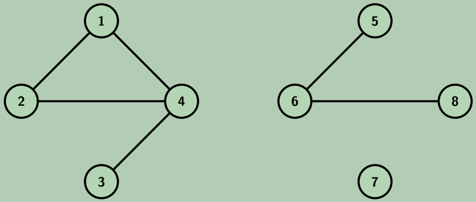

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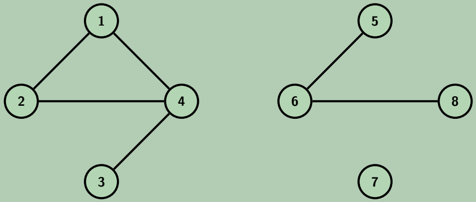

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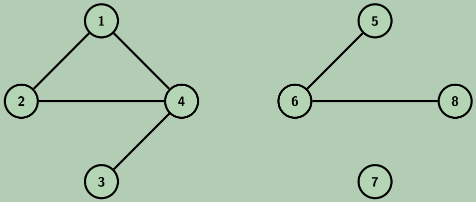

The graph on the left is connected. The graph on the right is not.

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Since these are the only two cases, we've shown that K_n is connected for every $n \in \mathbb{N} \setminus \{0\}$.

Crayola is the BEST

We say a graph $G = (V, E)$ is **2-colorable** iff we can use at most two colors (let's say red and black) to assign every $v \in V$ a color where $\{u, v\} \in E \implies u$ is a different color from v.

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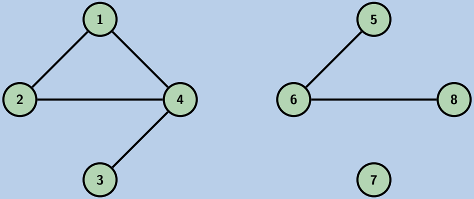

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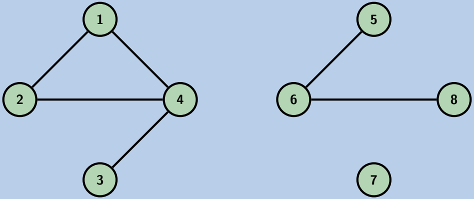

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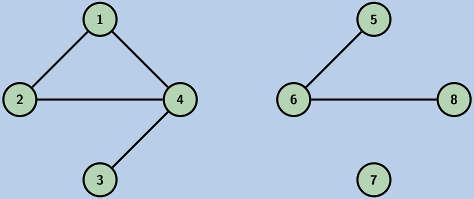

The graph on the left is not two-colorable. The graph on the right can be 2-colored by giving 5,7,8 red and 6 black (for example).

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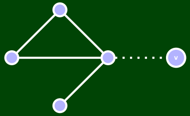

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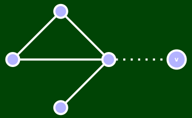

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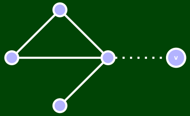

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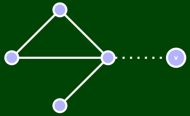

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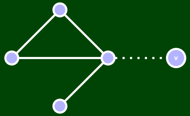

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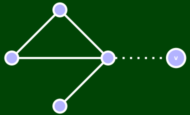

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Uh oh. . . something is wrong!

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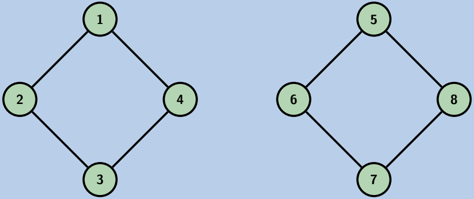

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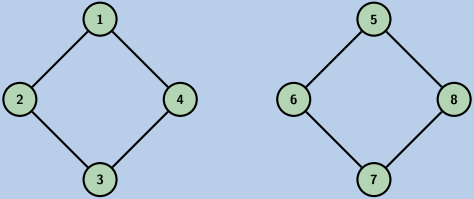

It clearly satisfies the property, and yet, it's disconnected.

Our Induction proof never covered this case, because you can't get to it by adding a single vertex at a time!

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So, how can we fix it?

Enter “Graph Induction”

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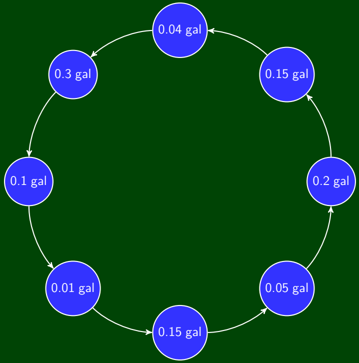
The big difference between Graph Induction and what we did before is that we didn't assume we could build a larger graph up from smaller graphs. Instead, we took a larger graph and found a way to invoke our IH.

Suppose that we have a car on a 1 meter circular track with n gas stations such that the total gas among all n gas stations is 1 gallon (where we use gas at a rate of 1 gallon per meter).



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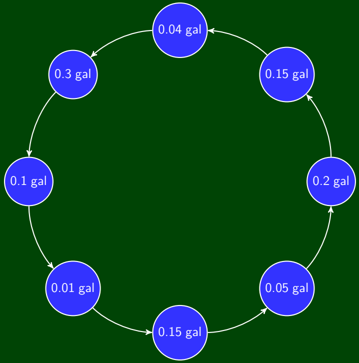

Prove that there is at least one gas station that we can start at with no gas such that we can make it all the way around the circle.

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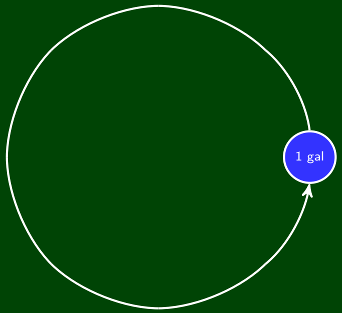

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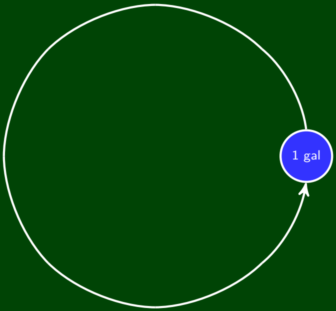
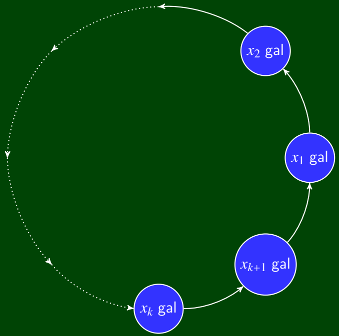


Since there is only one gas station, the entire gallon must be at it; so, we can make it all the way around.

Induction Hypothesis: Suppose that the claim is true for all possible tracks with k gas stations.
Induction Step: Suppose we have some track with $k+1$ gas stations:



We want to **remove** some gas station from the track, but which one should we remove?

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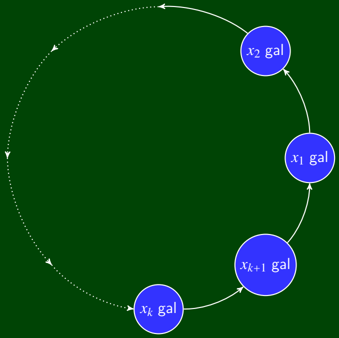

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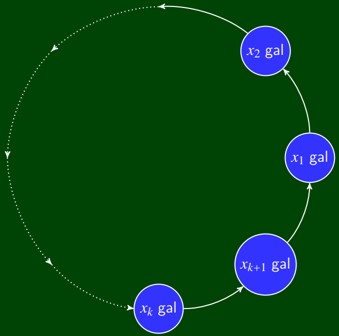
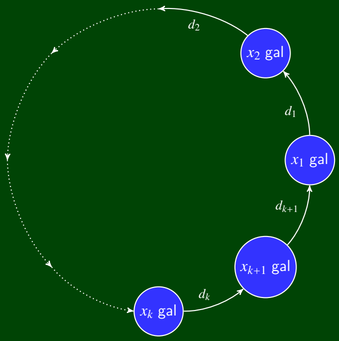
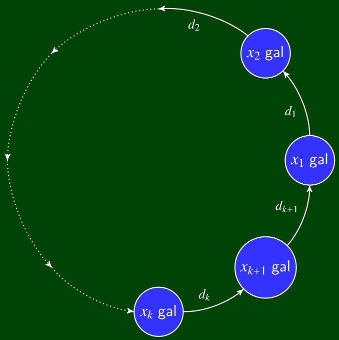


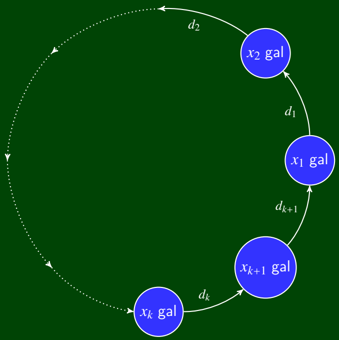
We want to **remove** some gas station from the track, but which one should we remove? The insight here is to find some gas station that will be useful once we eventually put it back in. **How about one that can get us across the gap?**

Finding x





Assume for the sake of contradiction that **none of the gas stations** had enough gas to get us to the next gas station.



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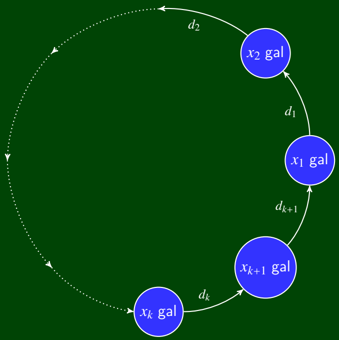

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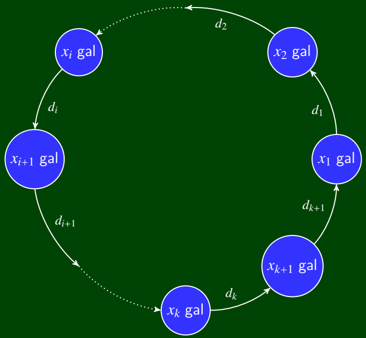
So, gas station i has enough to cross the gap!



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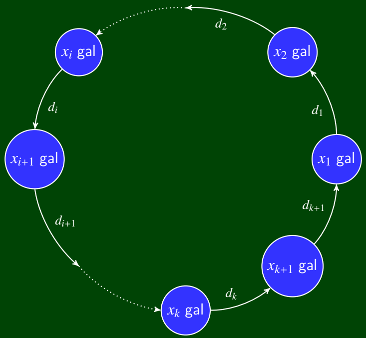

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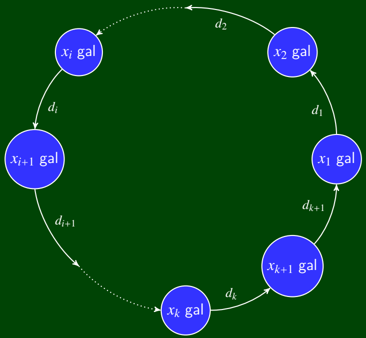

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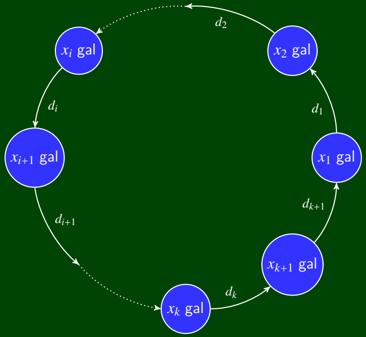

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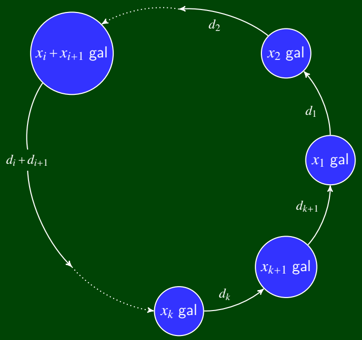
Question: What do we do with the gas at station $i+1$?

Invoking the IH

We give it to gas station i! So, once we've removed the $(i+1)$st station, the track looks like:



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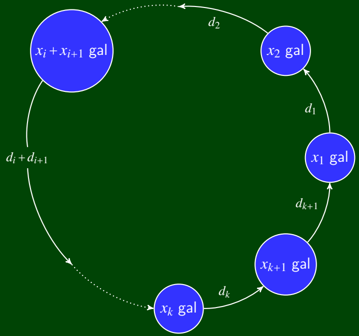

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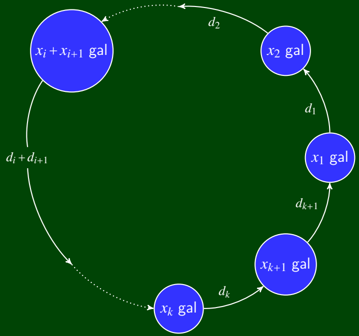
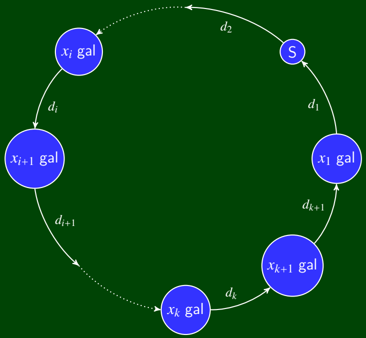


Since we now have a track with k gas stations, we can invoke the IH! It follows that with these k stations, there is a station s that can get us all the way around the track.

Now, we add station $i+1$ back in:



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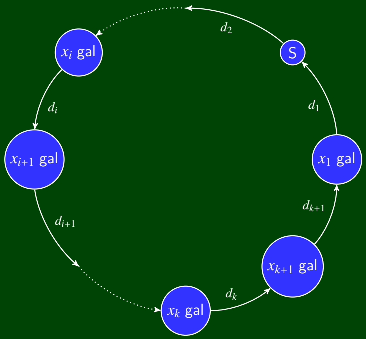

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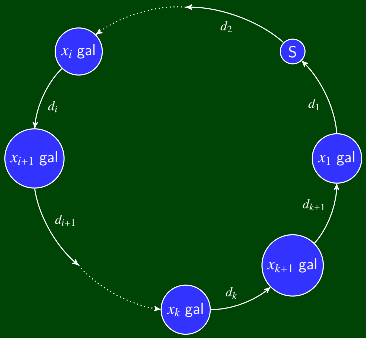

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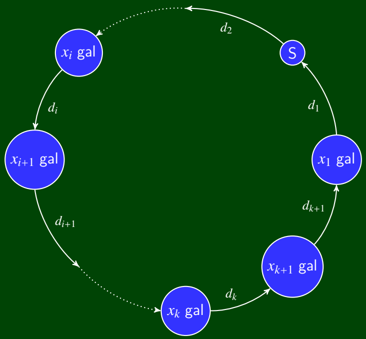

Note that by our IH, we know that we can get from station s to station i. We also know that we can get from station i to station $i+1$ with x_i gas, because we proved it earlier. Thus, we can get from s to i, from i to $i+1$, and from $i+1$ back to s. So, station s still works!

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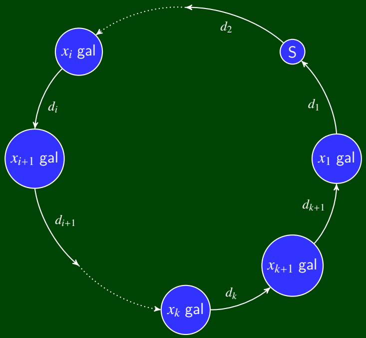

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Finally, since we showed the base case and the induction step, we know that the claim is true by induction.