CS 13

Mathematical Foundations of Computing

Probability

Some Code

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To drive the point home, we are going to only use probability as a way of **analyzing code with randomized components**.

We will always assume we have the following random primitives:

- **FlipCoin**(p) returns HEADS with probability p and TAILS otherwise.
- **RollDie**(N) returns $x \in [N]$ with probability $1/N$.

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Okay, enough of that. Let's do some problems!

One of the TAs has a gambling problem. They make all of the following bets with Adam:

- I bet a coin will come up `HEADS`.
- I bet a six-sided die will be even.
- I bet if each of us rolls a four-sided die, the sum will be 4.
- I bet if I roll a four-sided die, and you roll a die with that many sides, the result will be even.

Which of these bets has a more than 50% chance of paying off?

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  $\Pr(\text{even roll}) = 3/6 = 1/2$

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if ta_die + adam_die == 4:
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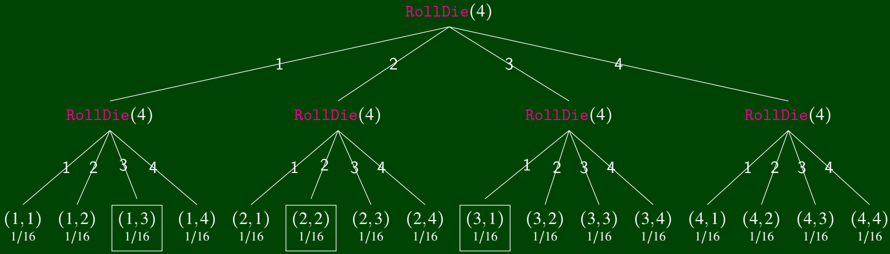

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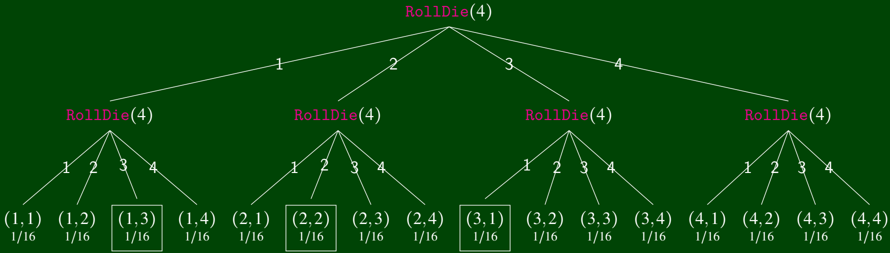

$$\Pr(\text{sum is 4}) = \Pr(\{(1,3),(2,2),(3,1)\}) = \frac{3}{16}$$

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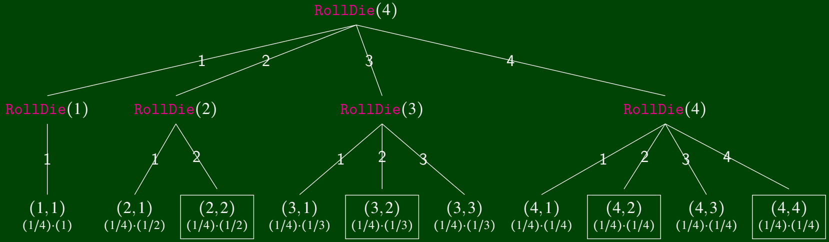

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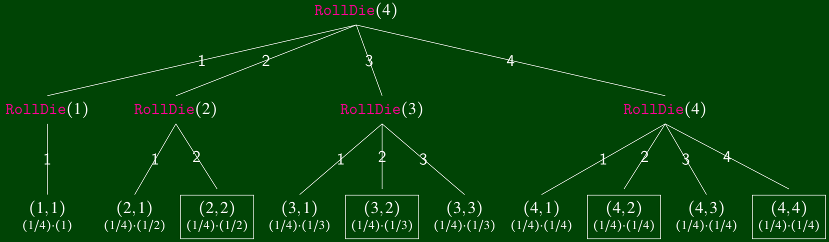

$$\Pr(\text{result is even}) = \Pr(\{(2,2),(3,2),(4,2),(4,4)\}) = \frac{1}{8} + \frac{1}{12} + \frac{1}{8} = \frac{1}{3}$$

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More generally:

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More generally: The probability of any sequence of 1313 flips of a fair coin is $\left(\frac{1}{2}\right)^{1313}$. They are all equally likely!

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It's actually only **23**! Let's prove it:
Let S be the sample space. We have 23 people each with one of 366 birthdays; so, the outcomes are length 23 strings of birthdays.
Let E be the event that at least two people have the same birthday. If we can calculate $|E|$ and $|S|$, we're done.

Since $|E|$ is hard to count, we count $|\overline{E}|$ instead! $\overline{E}$ is the event that no two people share a birthday. In other words, the event that the string has no duplicate entries. Using the Rule of Product, we see that

$$|\overline{E}| = (366)(365)\cdots(344)$$

We also know that $|S| = 366^{23}$ (also by the Rule of Product). Then, $\frac{|\overline{E}|}{|S|} = 0.494$ So,

$$\frac{|E|}{|S|} = 0.506$$

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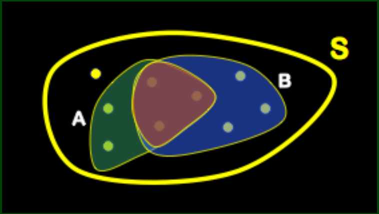

Think of $\Pr(A \mid B)$ as restricting the sample space to B and considering how often A occurs.

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Way 1: Let W be the event that the white die is 1 and S be the event that the sum is 7. We see that $\Pr(S) = \frac{6}{36}$ and $\Pr(S \cap W) = \frac{1}{36}$. Using the formula for conditional probability, we get $\Pr(W \mid S) = \frac{\frac{1}{36}}{\frac{6}{36}} = \frac{1}{6}$.

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Way 2: If we restrict our sample space to the six outcomes where the sum is 7, only one of them has the white die as 1. So, the probability we're looking for is $\frac{1}{6}$.

We have three bags with coins: one has two gold coins, one has two silver coins, and the third has one gold coin and one silver coin.

We randomly choose one of the bags, and we randomly select a coin from the bag. It turns out to be gold.
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Let G_1 be the event that the coin we chose first is gold. Let G_2 be the event that the second chosen coin is gold. The problem is asking us to find $\Pr(G_2 \mid G_1)$:

- $\Pr(G_1) = 3/6 = 1/2$ (three of the six coins are gold)
- $\Pr(G_1 \cap G_2) = 1/3$ (the coins must be in the gold-gold bag)
- $\Pr(G_2 \mid G_1) = \frac{\Pr(G_1 \cap G_2)}{\Pr(G_1)} = \frac{2}{3}$

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In Actuality: There are a total of 8 sessions of 98-000. Sandy met Pete at the first class which they both attended. For all future sessions, Pete took notes for Sandy, and she never showed up again.

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What is the probability that 1313 flips of a fair coin are all HEADS?

```
num_heads = 0
for i=1 to 1313:
   if FlipCoin(1/2) == HEADS:
      num_heads++
if num_heads == 1313:
   print "Yay!"
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$$\Pr\left(\bigcap_{i=1}^{1313} C_i\right) = \prod_{i=1}^{1313} \Pr(C_i) = \left(\frac{1}{2}\right)^{1313}$$

```
while FlipCoin() ≠ HEADS:
    print "Hello!"
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Does this code terminate?
Let E be the event that the code terminates.

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Let E be the event that the code terminates. Note that if E_i is the event that the code terminates on the ith run through the loop, we have:

$$E = \bigcup_{i=0}^{\infty} E_i$$

Then, we know $\Pr(E) = \Pr(\bigcup_{i=0}^{\infty} E_i) = \sum_{i=1}^{\infty} \Pr(E_i)$. Note that $\Pr(E_i)$ is the probability that the code does not terminate on the first $i-1$ iterations and does on the ith iteration. Then, $\Pr(E_i) = \left(\frac{1}{2}\right)^i$. So,

$$\Pr(E) = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i = \frac{1}{1-\frac{1}{2}} - 1 = 1$$