CS 13

Mathematical Foundations of Computing

Fancy Counting

$$1, 2, 3$$

Many of the questions we ask in counting are instances of the question:

> How many ways are there to place n balls into m bins?

We can make this question more interesting by varying the following:

- Are the balls distinguishable?
- Are the bins distinguishable?
- Any restrictions on how many balls in a bin? (exactly one, at least one, at most one, any number)

Let's start with the ones we already know. . . and work from there.

Indistinguishable Balls, Distinguishable Bins

How many ways are there to place n **indistinguishable** balls into m **distinguishable** bins?

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- **At most one ball.**
- **At least one ball.** ???
- **Any number of balls.** ???

How many ways are there to place n **indistinguishable** balls into m **distinguishable** bins?

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These last two are a bit harder. Let's try to make a counting argument...

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This means that there are n balls and $m-1$ dividers (which makes m bins!). The only step in our counting argument was to choose $m-1$ of the $n+m-1$ $\circ$'s to be dividers. So, there are $\binom{n+m-1}{m-1}$ ways to place these balls into bins.

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Sure. Take m of the balls and distribute them, one to each bin. Then, of the remaining $n-m$, give each bin any number of balls. There are

$$\binom{(n-m)+m-1}{m-1} = \binom{n-1}{m-1}$$

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Let's talk about "Inclusion-Exclusion"...

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More generally, Inclusion-Exclusion says:

$$\left|\bigcup_{i=0}^{n} A_i\right| = \sum_{i_1=0}^{n} |A_{i_1}| - \sum_{1 \le i_1 < i_2 \le n} |A_{i_1} \cap A_{i_2}| + \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|$$

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The next obvious question is "how do I use this?". The big thing to get about inclusion-exclusion is how to define the A_i's.
Let's do an example.

Consider the set

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Before we do anything else, we need to determine what A_i is supposed to be. If we have defined things correctly, then $S = A_1 \cup A_2 \cup \cdots \cup A_n$.

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Here are some proposals:

- A_i is the set of triples with i of the three coordinates equal to 1.
- A_i is the set of triples with the ith coordinate equal to 1.

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Here's why: (1) $A_1 \cup A_2 \cup A_3 = S$, (2) $|A_i|$ is easy to count!, (3) $\left|\bigcap_{i \in X} A_i\right|$ is easy to count!

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- $|A_1| = m\ell$, because we choose the first coordinate from 1 choice, the second coordinate from m choices, and the third from ℓ choices.
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Some options again. . .

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Ah! But we already know the total number (m^n), and removing $\bigcup_{i=0}^{m} A_i$ from our set leaves what we want!

Let S be the set of ways to place n **distinguishable** balls into m **distinguishable** bins.

A_i is the set of outcomes with no balls in the ith bin.

Here we go again...

$$\left|S \setminus \bigcup_{i=0}^{m} A_i\right| = m^n - \left(\sum_{k=1}^{m} (-1)^{k+1} \left(\sum_{1 \le i_1 < \cdots < i_k \le m} |A_{i_1} \cap \cdots \cap A_{i_k}|\right)\right)$$

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Consider one term of one of the inner summations:

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No! Remember, $A_{i_1} \cap \cdots \cap A_{i_k}$ is the set of outcomes where bins $i_1, i_2, \ldots, i_k$ are not hit. We know how to count this already: $|A_{i_1} \cap \cdots \cap A_{i_k}| = (m-k)^n$.

Now, considering the summation:

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And, finally, we have:

$$\left|S\setminus\bigcup_{i=0}^{m}A_i\right| = m^n - \left(\sum_{k=1}^{m}(-1)^{k+1}\left(\sum_{1\le i_1<\cdots<i_k\le m}\left|A_{i_1}\cap\cdots\cap A_{i_k}\right|\right)\right)$$
$$= m^n - \left(\sum_{k=1}^{m}(-1)^{k+1}\binom{m}{k}(m-k)^n\right)$$

$$=$$

Now, considering the summation:

$$\sum_{1\leq i_1<\cdots<i_k\leq m} |A_{i_1}\cap\cdots\cap A_{i_k}| = \sum_{1\leq i_1<\cdots<i_k\leq m} (m-k)^n$$
$$= \binom{m}{k}(m-k)^n$$

And, finally, we have:

$$\left|S\setminus\bigcup_{i=0}^{m} A_i\right| = m^n - \left(\sum_{k=1}^{m}(-1)^{k+1}\left(\sum_{1\leq i_1<\cdots<i_k\leq m}|A_{i_1}\cap\cdots\cap A_{i_k}|\right)\right)$$
$$= m^n - \left(\sum_{k=1}^{m}(-1)^{k+1}\binom{m}{k}(m-k)^n\right)$$
$$= \sum_{k=0}^{m}(-1)^k\binom{m}{k}(m-k)^n$$

How many ways are there to place n **indistinguishable** balls into m **distinguishable** bins?

- **Exactly one ball.** 1 if $n = m$, 0 otherwise
- **At most one ball.** $\binom{m}{n}$
- **At least one ball.** $\binom{n-1}{m-1}$
- **Any number of balls.** $\binom{n+m-1}{m-1}$

How many ways are there to place n **distinguishable** balls into m **distinguishable** bins?

- **Exactly one ball.** $m!$ if $n = m$, 0 otherwise
- **At most one ball.** $\frac{m!}{(m-n)!}$ if $m \geq n$
- **At least one ball.** $\sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n$
- **Any number of balls.** m^n