CS 13: Mathematical Foundations of Computing

Lecture 2 Exercises Solutions

Prime Numbers

Claim
Every integer $n \geq 2$ has a prime divisor.

Solution:

We go by strong induction.
Base Case: 2 is prime; so 2 is a prime divisor of 2.
Induction Hypothesis: Suppose the claim is true for all ℓ such that $\ell \leq k$ for some $k \geq 2$.
Induction Step: We go by cases:

- $k + 1$ is prime. Then, $k + 1$ is a prime divisor of $k + 1$.
- $k + 1$ is composite. Then, $k + 1 = ab$ for some $a, b < k + 1$ by definition of composite. Then, since $a < k + 1$, the IH applies, and a has a prime divisor. All divisors of a are also divisors of $k + 1$; so, $k + 1$ has a prime divisor.

Making Change

Claim

Suppose we have infinitely many 4 cent coins and 5 cent coins. Prove that we can make change for any $n \in \mathbb{N}$ where $n \geq 12$.

Solution:

We go by strong induction.
Base Cases:

- $12 = 4 * 3 + 0 * 5$
- $13 = 4 * 2 + 1 * 5$
- $14 = 4 * 1 + 2 * 5$
- $15 = 4 * 0 + 3 * 5$

Induction Hypothesis: Suppose the claim is true for all ℓ such that $\ell \leq k$ for some $k \geq 15$.
Induction Step: Note $k-3 = 4a+5b$ for some $a, b \in \mathbb{N}$ by the IH becaquse $k \geq 15$. Then, $k+1 = 4a+5b+4$. So, $k+1 = 4(a+1)+5b$ which is what we were trying to prove.