CS 13: Mathematical Foundations of Computing

Lecture 0 Exercises Solutions

Sets

Definition

The *power set* of a set X, denoted $\mathcal{P}(X)$, is defined as $\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$.

Consider the following examples:

- $\mathcal{P}(\varnothing) = \{\varnothing\}$
- $\mathcal{P}(\{a, b\}) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$

Claim

Prove that for all sets X and Y, $\mathcal{P}(X) \cup \mathcal{P}(Y) \cup \mathcal{P}(X \cap Y) \subseteq \mathcal{P}(X \cup Y)$.

Solution:

We show that for all sets X, Y. $\mathcal{P}(X) \cup \mathcal{P}(Y) \cup \mathcal{P}(X \cap Y) \subseteq \mathcal{P}(X \cup Y)$. Let $e \in \mathcal{P}(X) \cup \mathcal{P}(Y) \cup \mathcal{P}(X \cap Y)$. By the definition of union, it must be the case that $e \in \mathcal{P}(X)$ or $e \in \mathcal{P}(Y)$ or $e \in \mathcal{P}(X \cap Y)$. Each of these sets, $X, Y, X \cap Y$ is a subset of $X \cup Y$, so since e must be a subset of one of these sets, $e \subseteq X \cup Y$, so $e \in \mathcal{P}(X \cup Y)$.

Functions

Definition

A function $f : A \to B$ is *strictly monotone* iff it is either strictly increasing (for all $x, y \in A$, $x > y \implies f(x) > f(y)$) or it is strictly decreasing (for all $x, y \in A$, $x > y \implies f(x) < f(y)$).

Definition

A function $f : A \to B$ is *injective* iff for all $x, y \in A$, if $f(x) = f(y)$, then $x = y$.

Claim

Prove that if $f : A \to B$ is strictly monotone, then it is injective.

Solution:

We go by contrapositive. Let $x, y \in A$. Suppose $x \neq y$. Then, without loss of generality, let $x > y$. If f is strictly increasing, $f(x) > f(y)$; if f is strictly decreasing, $f(x) < f(y)$. In either case, $f(x) \neq f(y)$, as required.

Induction

Claim
For all $n \in \mathbb{N}$ and $x \in \mathbb{R}$ where $x > -1$, $(1+x)^n \geq 1 + nx$

Solution:

Let $x > -1$ be a real number. We go by induction to prove $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

Base Case. $(1+x)^0 = 1$ and $1 + 0x = 1$, so $(1+x)^0 \geq 1 + 0x$.

Induction Hypothesis. Suppose that the claim holds for $n = k$ for some $k \in \mathbb{N}$.

Induction Step. We show the claim for $n = k + 1$.

$$
\begin{aligned}
(1+x)^{k+1} &= (1+x)(1+x)^k && \text{[Factor out a } (1+x)\text{]} \\
&\geq (1+x)(1+kx) && \text{[Induction Hypothesis, and since } (1+x) \geq 0\text{]} \\
&= 1 + kx + x + kx^2 && \text{[Distribute]} \\
&\geq 1 + kx + x && [kx^2 \geq 0 \text{ since } k \geq 0 \text{ and } x^2 \geq 0] \\
&= 1 + (k+1)x && \text{[Factor]}
\end{aligned}
$$

So, the claim is true by induction.