CS 13: Mathematical Foundations of Computing

Strong Induction Solutions

Proof Technique Outline

Claim: [This is the claim you want to prove] for all $n \in \mathbb{N}$.
We go by induction on n to prove the claim.
Base Cases: When $n = 1$, the claim holds, because [prove the claim for $n = 1$ here].
[*You will probably need more than one base case! The way you figure out how many you need is the number of previous iterations you use in the Induction Step.* **Say there are x base cases**]
Induction Hypothesis: Suppose that the claim is true for all $k \leq \ell$ for some $\ell \in \mathbb{N}$ such that $k \in \mathbb{N}, \ell \geq x$.
Induction Step: [Use the Induction Hypothesis to prove the claim for $k + 1$.]

Since the Base Case and Induction Step hold, the claim holds for all $n \in \mathbb{N}$.

Proof Technique Example

Problem: Prove that if

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1 \\ a_n &= 2a_{n-1} + 3a_{n-2} \text{ for } n \geq 3 \end{aligned}$$

then $a_n = \frac{1}{2}(3^{n-1} - (-1)^n)$ for $n \in \mathbb{N}$.

Solution:

Claim: $a_{l-1} = \frac{1}{2}(3^{l-2} - (-1)^{l-1}$.
Base Cases:

- When $n = 1$, the claim is true, because $a_1 = 1$.
- When $n = 2$, the claim is true, because $a_2 = 1$.

Induction Hypothesis: Suppose that the claim is true for all $k \leq l$ such that $l \in \mathbb{N}$ such that $k \geq 1$, $l \geq 2$.
Induction Step: By the definition of a_n (where $n \geq 3$), we have: $a_{l+1} = 2a_l + 3a_{l-1}$. Now, note that by our IH, we know that $a_l = \frac{1}{2}(3^{l-1} - (-1)^l$ and $a_{l-1} = \frac{1}{2}(3^{l-2} - (-1)^{l-1})$.

$$\begin{aligned} &\text{Putting these together, we have} \quad a_{l+1} &&= 2\left(\tfrac{1}{2}(3^{l-1} - (-1)^l\right) + 3\left(\tfrac{1}{2}(3^{l-2} - (-1)^{l-1}\right) \\ &\text{Simplifying} &&= 3^{l-1} - (-1)^l + \tfrac{1}{2}\left(3^{l-1} - 3(-1)^{l-1}\right). \\ &\text{Combining} &&= \tfrac{1}{2}\left(2 \times 3^{l-1} + 3^{l-1} - 2(-1)^l - 3(-1)^{l-1}\right). \\ &\text{So} &&= \tfrac{1}{2}\left(3^l - (-1)^l(2-3)\right). \\ &\text{Finally} &&= \tfrac{1}{2}\left(3^l - (-1)^{l+1}\right) \end{aligned}$$

This is what we were trying to prove.

Several Questions/Answers

(a) What is ℓ?
ℓ is the largest value in the naturals that we are assuming the claim for. We are assuming $P(1), P(2), \ldots, P(\ell)$.

(b) Why is there no k in the Induction Step?
k is a variable that we are using to range from 1 to l. It is so that we can say "we are assuming $P(k)$ for $1 \leq k \leq \ell$" This is the same as saying "we are assuming $P(1), P(2), \ldots, P(\ell)$"

(c) How many base cases should there be?
Enough to ensure that you are covering every value in the naturals. If your Induction Step assumes that $\ell \geq x$, you should have base cases for $1, \ldots, x$.

(d) How large should ℓ be?
Large enough that everything you are talking about is well defined. For instance, in the above proof, we are saying that $a_{\ell+1} = 2a_\ell + 3a_{\ell-1}$ which is only true if $\ell + 1 \geq 3$ (which is the same as $\ell \geq 2$). So, you need $\ell \geq 2$.