CS 13: Mathematical Foundations of Computing

Lecture Exercises

Name:

E-mail:

0.

Functions on Trees

$$\begin{aligned}
\texttt{flip}(\texttt{Nil}) &= \texttt{Nil} \\
\texttt{flip}(\texttt{Tree}(x,\ L,\ R)) &= \texttt{Tree}(x,\ \texttt{flip}(R),\ \texttt{flip}(L))
\end{aligned}$$

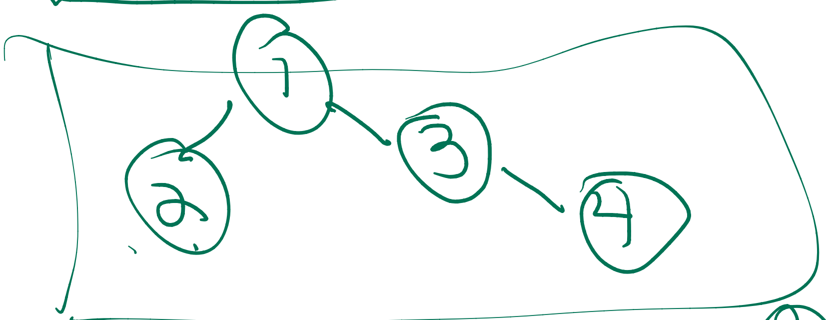
Claim

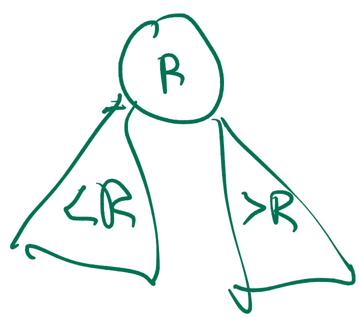
For all $T \in$ **Tree**,

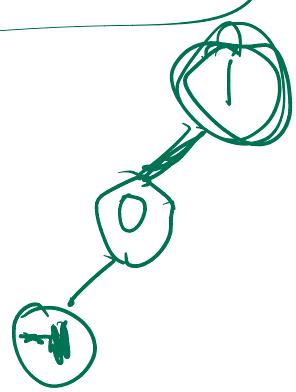
$$\texttt{flip}(\texttt{flip}(T)) = T$$

Tree = Nil | Tree(x, L, R) for $x \in \mathbb{Z}$, $L, R \in$ Tree

Tree(1, Tree(2, Nil, Nil), Tree(3, Nil, Tree(4, Nil, Nil)))







Case Nil:

case T(x, L, R)

1.

Functions on Trees

$$\begin{aligned} \texttt{full}(0) &= \texttt{Nil} \\ \texttt{full}(n) &= \texttt{Tree}(n,\ \texttt{full}(n-1),\ \texttt{full}(n-1)) \end{aligned}$$

$$\begin{aligned} \texttt{num\_leaves}(\texttt{Nil}) &= 1 \\ \texttt{num\_leaves}(\texttt{Tree}(x,\ L,\ R)) &= \texttt{num\_leaves}(L) + \texttt{num\_leaves}(R) \end{aligned}$$

Claim

For all $n \in \mathbb{N}$,

$$\texttt{num\_leaves}(\texttt{full}(n)) = 2^n$$

2.

Functions on Trees

$$\begin{aligned} \texttt{size}(\texttt{Nil}) &= 0 \\ \texttt{size}(\texttt{Tree}(x,\ L,\ R)) &= 1 + \texttt{size}(L) + \texttt{size}(R) \end{aligned}$$

$$\begin{aligned} \texttt{height}(\texttt{Nil}) &= 0 \\ \texttt{height}(\texttt{Tree}(x,\ L,\ R)) &= 1 + \texttt{max}(\texttt{height}(L), \texttt{height}(R)) \end{aligned}$$

Claim

For all $T \in$ **Tree**,

$$\texttt{size}(T) \leq 2^{\texttt{height}(T)} - 1$$

$$\text{size}(T(x, L, R)) = 1 + \text{size}(L) + \text{size}(R)$$

$$\leq 1 + 2^{\text{height}(L)} - 1 + 2^{\text{height}(R)} - 1 \text{ by IH}$$

$$2^{h(L)} \leq 2^{\max(h(L), h(R))}$$

$$\leq 1 + 2 \cdot 2^{\max(\text{height}(L), \text{height}(R))} - 2$$

This step is non-obvious

$$= -1 + 2^{1 + \max(h(L), h(R))}$$

$$= 2^{\text{height}(T(x, L, R))} - 1$$