CS 13

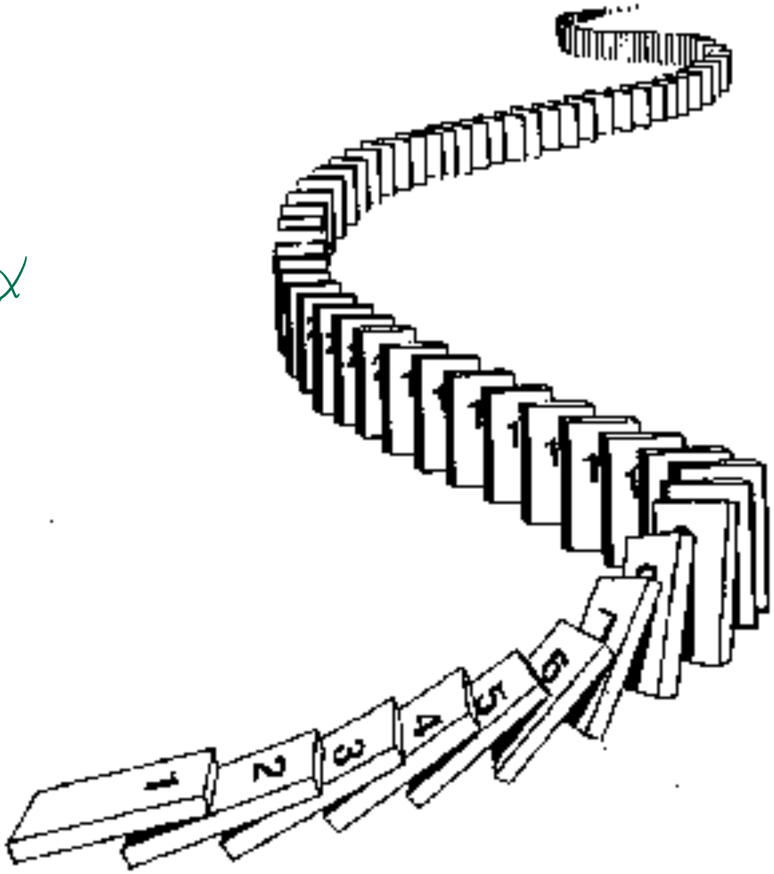
Mathematical Foundations of Computing

CS 13: Mathematical Foundations of Computing

Lecture 01: Induction

Administrivia:

- Late Tokens
- Lecturettes this week
- Reminder: use template & LaTex



Mathematical Induction

Method for proving statements about all natural numbers

- **A new proof technique!**
  - It only applies over the natural numbers
  - The idea is to **use** the special structure of the naturals to prove things more easily
- **Particularly useful for reasoning about programs!**

```
for(int i=0; i < n; n++) { … }
```

  - Show P(i) holds after i times through the loop

```
public int f(int x) {
    if (x == 0) { return 0; }
    else { return f(x – 1); }
}
```

  - f(x) = x for all values of x ≥ 0 naturally shown by induction.

So, make one!

Domain: Natural Numbers

$$P(0)$$
$$\forall\, k\, (P(k) \rightarrow P(k+1))$$

$$\therefore\ \forall\, n\, P(n)$$

Induction Is A Rule of Inference

Domain: Natural Numbers

$$P(0)$$
$$\forall\ k\ (P(k)\ \rightarrow\ P(k+1))$$

$$\therefore\ \forall\ n\ P(n)$$

How does this technique prove P(5)?

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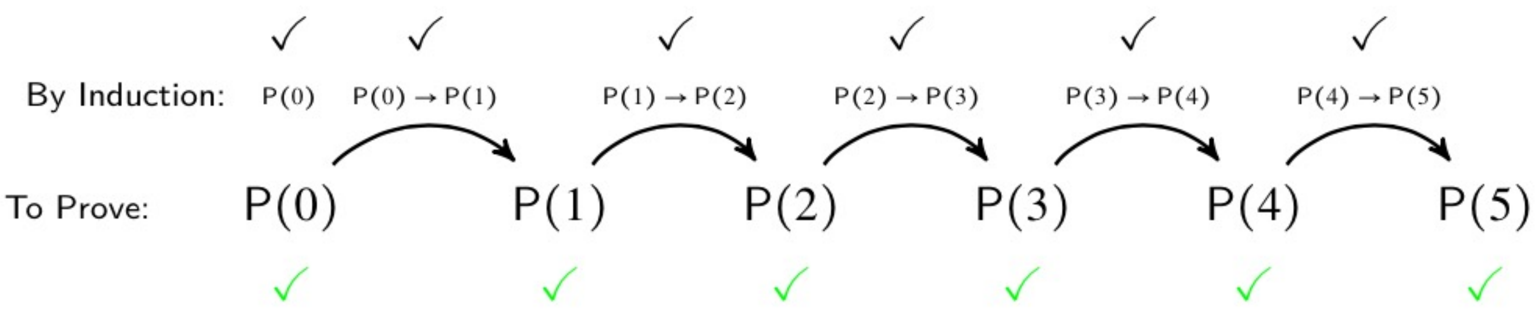
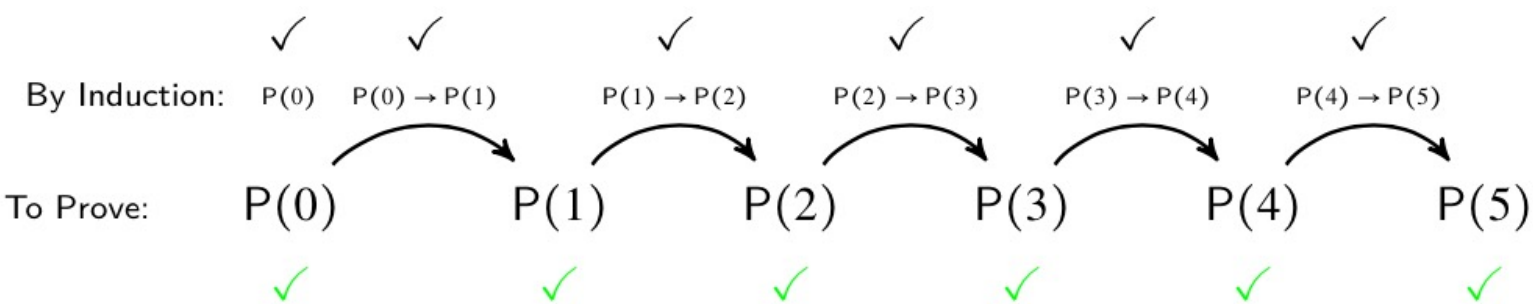


Induction Is A Rule of Inference

Domain: Natural Numbers

$$P(0)$$
$$\forall\, k\, (P(k) \rightarrow P(k+1))$$
$$\therefore\ \forall\, n\, P(n)$$

How does this technique prove P(5)?



First, we prove P(0).
Since P(n) → P(n+1) for all n, we have P(0) → P(1).
Since P(0) is true and P(0) → P(1), by Modus Ponens, P(1) is true.
Since P(n) → P(n+1) for all n, we have P(1) → P(2).
Since P(1) is true and P(1) → P(2), by Modus Ponens, P(2) is true.

5 Steps To Inductive Proofs In English

Proof:

1. "We will show that P(n) is true for every $n \geq 0$ by Induction."
2. "Base Case:" Prove P(0)
3. "Inductive Hypothesis:"

   Assume P(k) is true for some arbitrary integer $k \geq 0$"
4. "Inductive Step:" Want to prove that P(k+1) is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume P(k+1) !!)
5. "Conclusion: Result follows by induction"

Prove 1 + 2 + 4 + … + 2^n = 2^{n+1} – 1

Let $P(n)$ be "$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$". We go by induction on n.

<u>Base Case (n=0):</u>

Note that $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$, which is exactly $P(0)$.

<u>Induction Hypothesis:</u>

Suppose $P(k)$ is true for some $k \in \mathbb{N}$.

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

Let $P(n)$ be "$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$". We go by induction on n.

<u>Base Case (n=0):</u>

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<u>Induction Hypothesis:</u>

Suppose $P(k)$ is true for some $k \in \mathbb{N}$.

<u>Induction Step:</u>

We want to show $P(k+1)$. That is, we want to show:

$$\sum_{i=0}^{k+1} 2^i = 2^{(k+1)+1} - 1$$

$\iff 2^{(k+1)} + \sum = 2^{(k+1)+1} - 1$

$$\sum_{i=0}^{k+1} 2^i = \ldots = \ldots = \ldots = 2^{n+1} - 1.$$

One of these steps must use the IH.

So, the claim is true for all natural numbers by induction.

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

Let $P(n)$ be "$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$". We go by induction on n.

Base Case (n=0): Note that $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$, which is exactly $P(0)$.

Induction Hypothesis: Suppose $P(k)$ is true for some $k \in \mathbb{N}$.

Induction Step: We want to show $P(k+1)$. That is, we want to show: $\sum_{i=0}^{k+1} 2^i = 2^{(k+1)+1} - 1$

Note that $\sum_{i=0}^{k+1} 2^i =$

This is exactly $P(k+1)$. So, $P(k) \rightarrow P(k+1)$.

So, the claim is true for all natural numbers by induction.

We know (by IH)...

$\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

We're trying to get...

Our goal is to find a sub-expression of the left that looks like the left side of the IH.

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

Let $P(n)$ be "$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$". We go by induction on n.

Base Case (n=0): Note that $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$, which is exactly $P(0)$.

Induction Hypothesis: Suppose $P(k)$ is true for some $k \in \mathbb{N}$.

Induction Step: We want to show $P(k+1)$. That is, we want to show: $\sum_{i=0}^{k+1} 2^i = 2^{(k+1)+1} - 1$

$$\text{Note that } \sum_{i=0}^{k+1} 2^i = \left(\sum_{i=0}^{k} 2^i\right) + 2^{k+1} \quad \text{[Splitting the summation]}$$

$$= \left(2^{k+1} - 1\right) + 2^{k+1} \quad \text{[By IH]}$$

$$= \left(2^{k+1} + 2^{k+1}\right) - 1 \quad \text{[Assoc. of +]}$$

$$= \left(2(2^{k+1})\right) - 1 \quad \text{[Factoring]}$$

$$= 2^{k+2} - 1 \quad \text{[Simplifying]}$$

This is exactly $P(k+1)$. So, $P(k) \rightarrow P(k+1)$.

So, the claim is true for all natural numbers by induction.

Don't bother justifying the "obvious" steps. But make sure you say "by IH" somewhere.

We know (by IH)...

$$\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$$

We're trying to get...

$$\sum_{i=0}^{k+1} 2^i = 2^{(k+1)+1} - 1$$

Our goal is to find a sub-expression of the left that looks like the left side of the IH.

Prove 1 + 2 + 3 + … + n = n(n+1)/2

Let $P(n)$ be "$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$". We go by induction on n.

Base Case (n=0):

Induction Hypothesis:

Induction Step:

We know (by IH)…

We're trying to get…

Our goal is to find a sub-expression of the left that looks like the left side of the IH.

This is exactly $\mathrm{P}(k+1)$. So, $\mathrm{P}(k) \rightarrow \mathrm{P}(k+1)$.

So, the claim is true for all natural numbers by induction.

Prove 1 + 2 + 3 + … + n = n(n+1)/2

Let $P(n)$ be "$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$". We go by induction on n.

Base Case (n=0): Note that $\sum_{i=0}^{0} i = 0 = \frac{0(0+1)}{2}$, which is exactly $P(0)$.

Induction Hypothesis: Suppose P(k) is true for some k ∈ ℕ.

Induction Step: We want to show P($k+1$). That is, we want to show: $\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$

Note that $\sum_{i=0}^{k+1} i = \left(\sum_{i=0}^{k} i\right) + (k+1)$ **[Splitting the summation]**

$= \left(\frac{k(k+1)}{2}\right) + (k+1)$ **[By IH]**

$= (k+1)\left(\frac{k}{2} + 1\right) = (k+1)\left(\frac{k+2}{2}\right)$ **[Algebra]**

$= \frac{(k+1)(k+2)}{2}$ **[Algebra]**

This is exactly P($k+1$). So, P(k) → P($k+1$).

So, the claim is true for all natural numbers by induction.

We know (by IH)…

$\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$

We're trying to get…

$\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$

Our goal is to find a sub-expression of the left that looks like the left side of the IH.

Prove $3^n \geq n^2$ for all $n \geq 3$.

Let $P(n)$ be "$3^n \geq n^2$". We go by induction on n. $3^3 = 27 \geq 9 = 3^2$

Base Case (n=3): $3^3 \geq 3^2 \iff 27 \geq 9$

Induction Hypothesis: Suppose the claim is true for some $n \geq 3$

Induction Step: We want to show P(*k* + 1). the claim for $k+1$

Note that $3^{k+1} =$

$\geq$

$= (k+1)^2$

This is exactly P(*k* + 1). So, P(*k*) → P(*k* + 1).

So, the claim is true for all n ≥ 3 by induction.

We know (by IH)...

We're trying to get...

Strong Induction

$P(0)$

$\forall k\ \Big(\big(P(0) \land P(1) \land P(2) \land \cdots \land P(k)\big) \to P(k+1)\Big)$

$\therefore \forall n\ P(n)$

Strong Induction English Proof

1. By induction we will show that $P(n)$ is true for every $n \geq 0$
2. Base Case: Prove $P(0)$
3. Inductive Hypothesis:
   Assume that for some arbitrary integer $k \geq 0$, $P(j)$ is true for every j from 0 to k
4. Inductive Step:
   Prove that $P(k+1)$ is true using the Inductive Hypothesis (that $P(j)$ is true for all values $\leq k$)
5. Conclusion: Result follows by induction

Every $n \geq 2$ can be expressed as a product of primes.

Let $P(n)$ be "$n = p_0 p_1 \cdots p_j$, where $p_0, p_1, \ldots, p_j$ are prime."

We go by strong induction on n.

Base Case (n=2):

Induction Hypothesis: Suppose the claim for all $2 \leq \ell < k$ for some $k \geq 2$

Induction Step: We go by cases.

Case k is prime:

Case k is composite:

$k = ab$ for some $k > a, b > 1$

IH

We know (by IH)…

All numbers smaller than k can be expressed as a product of primes.

We're trying to get…

k can be expressed as a product of primes.